MATH4240: Stochastic Processes Tutorial 5

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Examples on birth and death chain

Let X_n, $n \geq 0$ be an irreducible birth and death chain on nonnegative integers with birth probability $p_x > 0$ for $x \geq 0$ and death probability $q_y > 0$ for $y \geq 1$. Set $\gamma_0 = 1$ and $\gamma_y = \dfrac{q_1 \cdots q_y}{p_1 \cdots p_y}$ for $y \geq 1$. Recall that an irreducible birth and death chain on $\{0, 1, 2, \dots\}$ is recurrent if and only if

$$\sum_{x=1}^{\infty} \gamma_x = \infty$$

Examples on birth and death chain

Consider the birth and dearh chain on $\{0, 1, 2, \dots\}$ defined by

$$p_x = \frac{x+2}{2(x+1)}$$

and

$$q_x = \frac{x}{2(x+1)}.$$

Then, the chain is transient since $\frac{q_x}{p_x} = \frac{x}{x+2}$, and it follows that

$$\gamma_x = \frac{q_1 \dots q_x}{p_1 \dots p_x} = \frac{1 \cdot 2 \cdot \dots \cdot x}{3 \cdot 4 \cdot \dots \cdot (x+2)} = \frac{2}{(x+1)(x+2)} = 2(\frac{1}{x+1} - \frac{1}{x+2}).$$

Examples on birth and death chain

Thus,

$$\begin{aligned}\sum_{x=1}^{\infty}\gamma_x &= 2\sum_{x=1}^{\infty}(\frac{1}{x+1}-\frac{1}{x+2}) \\ &= 2(\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\dots) \\ &= 2\cdot\frac{1}{2}=1.\end{aligned}$$

We conclude that the chain is transient.

Examples on birth and death chain

Now, $\gamma_x = 2(\frac{1}{x+1} - \frac{1}{x+2})$. Hence,

$$P_x(T_a < T_b) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y} = \frac{2(\frac{1}{x+1} - \frac{1}{b+1})}{2(\frac{1}{a+1} - \frac{1}{b+1})} = \frac{(a+1)(b-x)}{(x+1)(b-a)}.$$

Recall that

$$P_x(T_a < T_b) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y}, \quad a < x < b.$$

Thus,

$$P_x(T_0 < T_n) = \frac{\sum_{y=x}^{n-1} \gamma_y}{\sum_{y=0}^{n-1} \gamma_y} = 1 - \frac{\sum_{y=0}^{x-1} \gamma_y}{\sum_{y=0}^{n-1} \gamma_y},$$

for $0 < x < n$.

Examples on birth and death chain

Note that for $x > 0$, $1 \leq T_{x+1} < T_{x+2} < \cdots$. Hence $\{T_0 < T_n\}_{n=1}^{\infty}$ forms a nondecreasing sequence of events. By continuity of the probability, we have for $x \geq 1$,

$$\begin{aligned}
\rho_{x0} &= P_x(T_0 < \infty) \\
&= P_x\left(\bigcup_{n=1}^{\infty}\{T_0 < T_n\}\right) \\
&= \lim_{n\to\infty} P_x(T_0 < T_n) \\
&= 1 - \lim_{n\to\infty} \frac{\sum_{y=0}^{x-1}\gamma_y}{\sum_{y=0}^{n-1}\gamma_y}.
\end{aligned}$$

Thus,

$$\rho_{x0} = 1 - \frac{\sum_{y=0}^{x-1}\gamma_y}{\sum_{y=0}^{\infty}\gamma_y} = \frac{\sum_{y=x}^{\infty}\gamma_y}{\sum_{y=0}^{\infty}\gamma_y} = \frac{\frac{2}{x+1}}{2} = \frac{1}{x+1}.$$

Remark. $q_x < p_x$ for all x does not imply the chain is transient. For example, one may take $\gamma_x = 1/2x$ by choosing $q_1/p_1 = 1/2$ and $q_n/p_n = (n-1)/n$ for $x \geq 2$. Then,

$$\sum_{x=0}^{\infty} \gamma_x = \infty$$

and thus the chain is recurrent.

Examples on birth and death chain

On the contrary, given an irreducible birth and death chain on nonnegative integers, if $p_x \leq q_x$ for $x \geq 1$, then

$$\sum_{y=0}^{\infty} \gamma_y = 1 + \sum_{y=1}^{\infty} \frac{q_1 \cdots q_y}{p_1 \cdots p_y} \geq 1 + \sum_{y=1}^{\infty} 1^y = \infty.$$

This implies that $\rho_{10} = 1$. By one-step argument, we have

$$\rho_{00} = P(0,0)\rho_{00} + P(0,1)\rho_{10} = r_0\rho_{00} + p_0.$$

Since $p_0 + r_0 = 1$ and $p_0 > 0$, we have $\rho_{00} = 1$, that is, state 0 is recurrent. As the chain is irreducible, it is recurrent.

Examples on branching chain

Consider a branching chain such that $f(1) < 1$.
If $f(0) > 0$, then for any $x > 0$,

$$P(x, 0) = f(0)^x > 0.$$

Since 0 is absorbing, any positive x is transient.
If $f(0) = 0$, then X_n is nondecreasing, that is, $\rho_{xy} = 0$ for $x > y$.
Moreover, for $x > 0$,

$$\rho_{xx} = P(x, x) = f(1)^x < 1.$$

Hence any positive x is transient.

Examples on branching chain 2

Consider a branching chain with $f(0) = f(3) = 1/2$. The mean number of offspring of one given particle is $\mu = 3/2 > 1$. Hence the extinction probability ρ is the root of the equation

$$\frac{1}{2} + \frac{1}{2}t^3 = t$$

lying in $[0, 1)$. We can rewrite this equation as

$$(t-1)(t^2 + t - 1) = 0.$$

This equation has three roots, namely, 1, $\frac{-1+\sqrt{5}}{2}$, and $\frac{-1-\sqrt{5}}{2}$. Consequently, $\rho = \frac{-1+\sqrt{5}}{2}$.

Examples on branching chain 3

Consider a branching chain. We would like to show $E_x[X_n] = x\mu^n$.
The conclusion holds trivially for $x = 0$. Now, for $x \geq 1$,

$$\sum_y yP(x,y) = E_x(X_1) = E(\xi_1 + \xi_2 + \cdots + \xi_x) = xE(\xi_1) = \mu x.$$

Now,

$$\begin{aligned} E_x(X_n) &= \sum_{y \in \mathcal{S}} yP(X_n = y) \\ &= \sum_{y \in \mathcal{S}} y\Big(\sum_{x \in \mathcal{S}} P(x,y)P(X_{n-1} = x)\Big) \\ &= \sum_{x \in \mathcal{S}} P(X_{n-1} = x)\Big(\sum_{y \in \mathcal{S}} yP(x,y)\Big) \\ &= \mu \sum_{x \in \mathcal{S}} xP(X_{n-1} = x) = \cdots = x\mu^n. \end{aligned}$$

Examples on queuing chain

We show that the chain is irreducible if and only if $f(0) > 0$ and $f(0) + f(1) < 1$.

If $f(0) = 0$, then $P(x, x-1) = f(0) = 0$ for $x \geq 1$. That implies $\rho_{xy} = 0$ for $x > y \geq 0$. Hence the chain is not irreducible.

If $f(0) + f(1) = 1$, then $P(x, y) = f(y - x + 1) = 0$ for $1 \leq x < y$. That implies $\rho_{xy} = 0$ for $1 \leq x < y$. Hence the chain is not irreducible.

This proves the "only if" part.

Examples on queuing chain

Now, suppose $f(0) > 0$ and $f(0) + f(1) < 1$.
For $x > y \geq 0$,

$$\rho_{xy} \geq P(x, x-1)P(x-1, x-2)\cdots P(y+1, y) = (f(0))^{x-y} > 0.$$

Since $f(0) + f(1) < 1$, there exists $x_0 \geq 2$ such that $f(x_0) > 0$. Then for $n \geq 0$,

$$\begin{aligned}\rho_{0,x_0+n(x_0-1)} &\geq P(0, x_0)P(x_0, x_0 + (x_0 - 1)) \\ &\quad P(x_0 + (x_0 - 1), x_0 + 2(x_0 - 1))\cdots \\ &\quad P(x_0 + (n-1)(x_0 - 1), x_0 + n(x_0 - 1)) \\ &= f(x_0)^{n+1} > 0.\end{aligned}$$

Now for any states x, y, there exists n such that $x_0 + n(x_0 - 1) > y$. Since x leads to 0, 0 leads to $x_0 + n(x_0 - 1)$, $x_0 + n(x_0 - 1)$ leads to y, x also leads to y. Hence the chain is irreducible. This proves the "if" part.